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Formation of asymmetric states of spiral waves in oscillatory media

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The stability of the symmetric two-spiral state against asymmetric perturbations is investigated. It is shown that the bound states are always unstable against a spontaneous breaking of symmetry, which leads to the formation of states with one dominant spiral. The existence and the stability of lattices of topological defects are discussed.

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The complex Ginzburg-Landau equation (CGLE)

$$\frac{\partial a}{\partial t} = a + (1 + ib)\Delta a - (1 + ic)|a|^2 a \qquad (1)$$

plays the role of a normal form in the vicinity of a supercritical transition to an oscillatory state in spatially extended systems and is thus very general. Here the complex field a describes the amplitude and phase of the modulations of the pattern [1–3]. In a recent paper [4] we have presented a detailed investigation of the asymptotic interaction of spirals. Spiral solutions have short-range interaction for $b - c \neq 0$ [5, 6]. Here interaction manifests itself in a motion of each spiral. The resulting velocity has a radial (along the line connecting the spiral cores) and a tangential component.

In Ref. [4] we considered only symmetric spiral pairs where the two spirals are equivalent (equally or oppositely charged). Approximate solutions were constructed by starting with isolated spirals, each one restricted to the half space filled by its emitted wave and moving with (small) velocities to be determined. The remaining distortions were assumed to be small and essentially time independent and could be determined from the linearized problem with boundary conditions that took into account the neighboring spiral. Once a system of linearly independent solutions for the distortions was determined numerically and used to match the boundary conditions the velocity versus distance relation was obtained analytically. The analysis showed that for sufficiently large values of $|b - c|$, i.e., for $|(c - b)/(1 + bc)| > c_{cr} \approx 0.845$, the interaction is oscillatory, leading to symmetric bound states. For smaller values of $|b - c|$ one always appears to have asymptotic repulsion turning into attraction at smaller distances for oppositely charged spirals.

There is numerical evidence that symmetric bound states after a sufficiently long evolution spontaneously break the symmetry and one spiral begins to dominate, pushing away other spirals [7]. In fact, at least in the convectively unstable but absolutely stable range, a symmetry-broken state is produced directly from random initial conditions [8]. To understand the symmetry-breaking instability one may consider the perturbation of the frequency ω of the waves emitted by each spiral, caused by the interaction with the other spiral. Indeed, from the analysis of CGLE it is known that the shock (or sink) where two waves with different frequencies ω_i collide moves in the direction of smaller frequency, which means that after a sufficiently long time only the larger frequency [or equivalently the larger wave number because of the dispersion relation $\omega = -c(1 - k^2) - bk^2$] dominates in a bounded system. The velocity of the motion is given by $v_f \sim (c - b)(k_1 + k_2) = \frac{1}{2}(v_{g1} + v_{g2})$, where $v_g = d\omega/dk$ is the group velocity of a plane-wave state. Therefore, if, due to the interaction, the frequencies of rotation of the spirals become different, one can expect a breaking of the symmetry of the system. This effect should be very important in the late stage of spiral evolution.

In this article we show that the two cases characterizing the behavior of the interaction in fact also characterize the symmetry breaking. Thus symmetric bound states are in fact (weakly) unstable with respect to asymmetric perturbations, and there exist stable asymmetric lattices of topological defects. In the case of asymptotic repulsion one has symmetric lattices of topological defects.

The (one-armed or singly-charged) isolated spiral solution of Eq. (1) is of the form

$$\tilde{a}(r,\theta) = F(r)\exp[i(\omega t \pm \theta + \psi(r) + \varphi)] \qquad (2)$$

and satisfies the following equations for the real functions $F(r)$ and $\psi(r)$:

$$\Delta_r F - \frac{1}{r^2}F - (\psi')^2 F - b[(\Delta_r\psi)F + 2\psi' F'] + F - F^3 = 0,$$
$$b\left(\Delta_r F - \frac{1}{r^2}F - (\psi')^2 F\right) + (\Delta_r\psi)F + 2\psi' F' - \omega F - cF^3 = 0, \qquad (3)$$

where (r,θ) are polar coordinates, $\Delta_r = \partial_r^2 + \frac{1}{r}\partial_r$ and primes denote derivatives with respect to r, and φ is some constant. The functions F and ψ have the following asymptotic behavior:

$$F(r) \to \sqrt{1-k^2}\ ,\quad \psi'(r) \to k,\quad r \to \infty, \qquad F(r) \sim r\ ,\quad \psi'(r) \sim r\ ,\quad r \to 0, \tag{4}$$

with $\omega = -bk^2 - c(1-k^2)$. The constant k is the asymptotic wave number of the waves emitted by the spiral, which is determined uniquely for given b, c. In general k has to be determined numerically (see, e.g., [9, 10]).

Due to the interaction with other spirals or with a boundary, the spiral core moves with some velocity $\mathbf{v}$. Also the phase φ becomes slowly varying in time. Inside its region of influence bounded by the shock structures the perturbed spiral solution can be written in the form

$$a(r,\theta) = [F(r) + W(r,\theta,t)]\exp\{i[\omega t + \theta + \psi(r) + \varphi(t)]\}, \tag{5}$$

where W is the correction to the unperturbed spiral solution and (r,θ) are now the coordinates comoving with the velocity v (W can be complex). Clearly $\partial_t\varphi$, which was not included in Ref. [4], describes the correction to the frequency of rotation of the individual spiral. The velocity of the drift $\mathbf{v}$ and the frequency change $\partial_t\varphi$ have to be chosen from the condition that the correction W remain small and slowly time dependent. (This description of the relevant time dependence induced by interaction in terms of a velocity and frequency change is strictly valid only in the range $vr \ll 1$. However, for larger distances the influence of perturbations can be neglected altogether as long as there are no other sources of waves.)

We now consider two spirals located on the x axis at $\pm X$. The problem of the interaction of two symmetric oppositely charged or equally charged spirals, where the shock is located at $x = 0$, is equivalent to the problem of the interaction of one spiral with a plane boundary with appropriate boundary conditions. To construct a slightly asymmetric two-spiral state, we have to suppose that the phases $\varphi_{1,2}$ are different (the other quantities except W remain the same for the two spirals). Then the position X_0 of the shock between the spirals will depend on the phase difference $\varphi_1 - \varphi_2$ and can be determined from the condition of continuity of the field:

$$a_1 = a_2 \quad \text{for } x = X_0,\quad y = 0. \tag{6}$$

The position of the shock can be determined easily in the limit of large separation. Substituting (5) into (6) we obtain at leading order of expansion in $1/X$ the condition of continuity in the form

$$k(X - X_0) + \varphi_1 = k(X + X_0) + \varphi_2, \tag{7}$$

and therefore

$$X_0 = \varphi/2k, \tag{8}$$

where $\varphi = \varphi_1 - \varphi_2$. The matching of the solutions at the position of the shock requires

$$\partial a_{1,2}/\partial x = 0 \quad \text{for } x = X_0,\quad y = 0. \tag{9}$$

(In a first approximation we can neglect the curvature of the shock line.)

Allowing for nonzero values of φ the analysis of Ref. [4] gives, for oppositely charged spirals in the limit $X \gg \varphi/(2k)$, the following equations of motion:

$$\begin{aligned}
v_{1x} &= \mathrm{Im}\left(\frac{-k\sqrt{1-k^2}\exp\{-p[X-\varphi/(2k)]\}}{\delta C_y\sqrt{2\pi pX}}X^{-\mu}\right)\Big/\mathrm{Im}(C_x/C_y),\\
v_{1y} &= \mathrm{Re}\left(\frac{-k\sqrt{1-k^2}\exp\{-p[X-\varphi/(2k)]\}}{\delta C_y\sqrt{2\pi pX}}X^{-\mu}\right) - v_{1x}\mathrm{Re}(C_x/C_y),\\
\partial_t\varphi &= 2\mathrm{Im}\left[\frac{-k\sqrt{1-k^2}\exp(-pX)}{\delta C_0\sqrt{2\pi pX}}X^{-\mu}\sinh\left(\frac{p\varphi}{2k}\right)\right]\Big/\mathrm{Im}(C_{10}/C_0) \approx \zeta\varphi
\end{aligned} \tag{10}$$

for $p\varphi/k \ll 1$. Analogous equations hold for the velocity of the second spiral. Here p is the decay rate of small perturbations around the isolated spiral solution. As shown in Ref. [4], p is complex for $|(c-b)/(1+bc)| > c_{cr} \sim 0.845$ ("oscillatory range") and real otherwise ("monotonic range"). $C_{x,y}, C_{0,1}$ are constants obtained from the numerical solution of the linearized problem for the first and zero harmonic of the perturbation, and μ, δ are functions of k, p given in Ref. [4]. Symmetric bound states correspond to the case $\varphi = 0$, $v_x = 0$, and $v_{1y} = v_{2y}$. They exist only in the oscillatory range. The constant ζ determined numerically turns out to be positive for the (first) symmetric bound state (see Fig. 1), so that the state is unstable with respect to φ. Hence the frequencies of rotation (and therefore the underlying wave numbers) become different, which means a breaking of the symmetry of the two-spiral solution. However, in fact, ζ is very small, so the symmetric bound states can be rather long lived. The symmetry breaking was also observed for other parameters, including $b \neq 0$, inside the oscillatory range.

As a result of the symmetry breaking in a bounded system only one "free" spiral will remain, whereas the other spiral is pushed away to the boundary. Depending on the boundary conditions the second spiral will finally either annihilate at the boundary (nonflux boundary con-

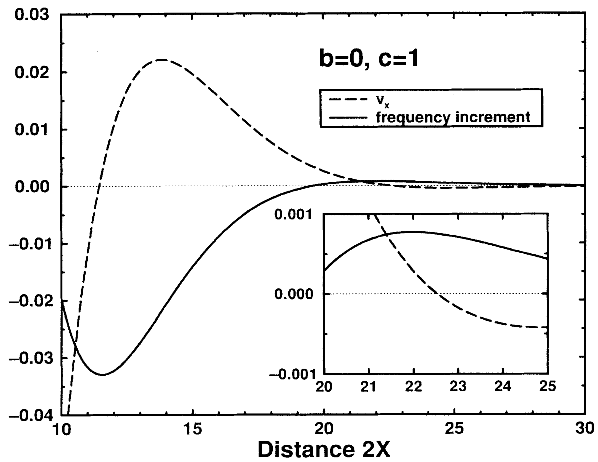

FIG. 1. The dependence of radial velocity v_x and the frequency increment $\zeta = d(\partial_t\varphi)/d\varphi$ at $\varphi = 0$ for $b = 0$, $c = 1$.

ditions, i.e., zero normal derivative on the boundary), or, with periodic boundary conditions, the defect will persist for topological reasons, but will be reduced to its core and enslaved in the corner of the shock structure of the free spiral (see Fig. 2). The last case leads to an asymmetric lattice of topological defects, which appears to be stable in the oscillatory case. We have tested the stability of such lattices in simulations of up to 4×4 cells.

The analysis can be simplified considerably for the case $|c - b| \to 0$ and $|(c-b)k|X \gg 1$. The matching with the outer solution can then be done analytically (see, e.g., [6, 11]) using a phase diffusion equation. From the analysis the equations for the frequency can be inferred in explicit form for $|(c-b)k|X \gg 1$,

$$\partial_t\varphi = -2(1+b^2)k^2\sqrt{\left|\frac{\pi c'}{kX}\right|}\exp(-2|c'kX|)\sinh(|c'|\varphi), \tag{11}$$

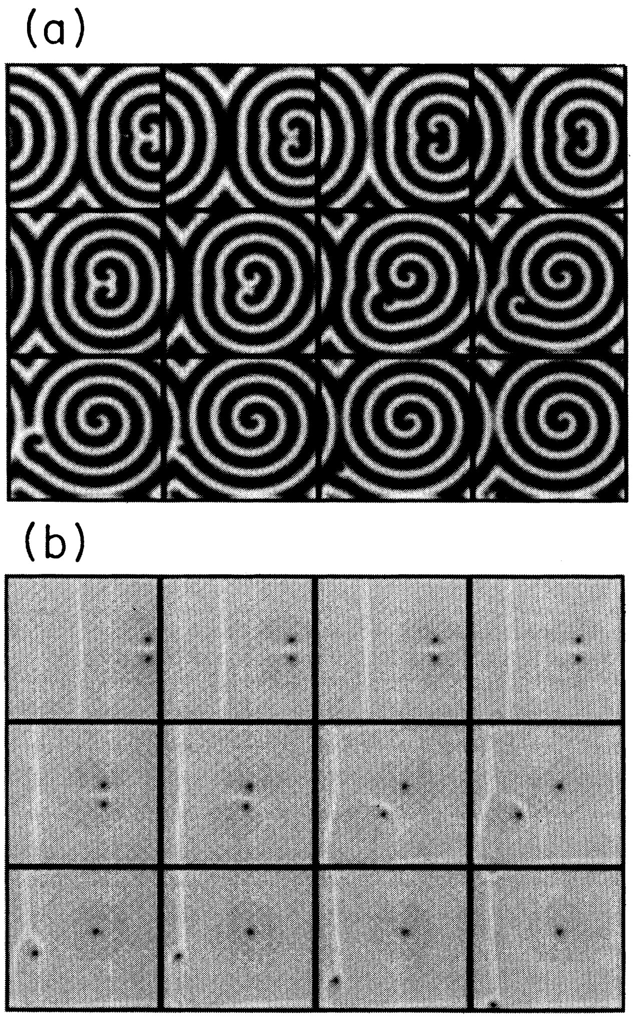

FIG. 2. Snapshots of the evolution of an oppositely charged spiral pair into a stable antisymmetric state; $b = 0$, $c = 0$, $L_x = L_y = 100$, periodic boundary conditions. Time runs from top left to bottom right, $\Delta t = 500$. Real part (a) and modulus (b) are shown in gray scale.

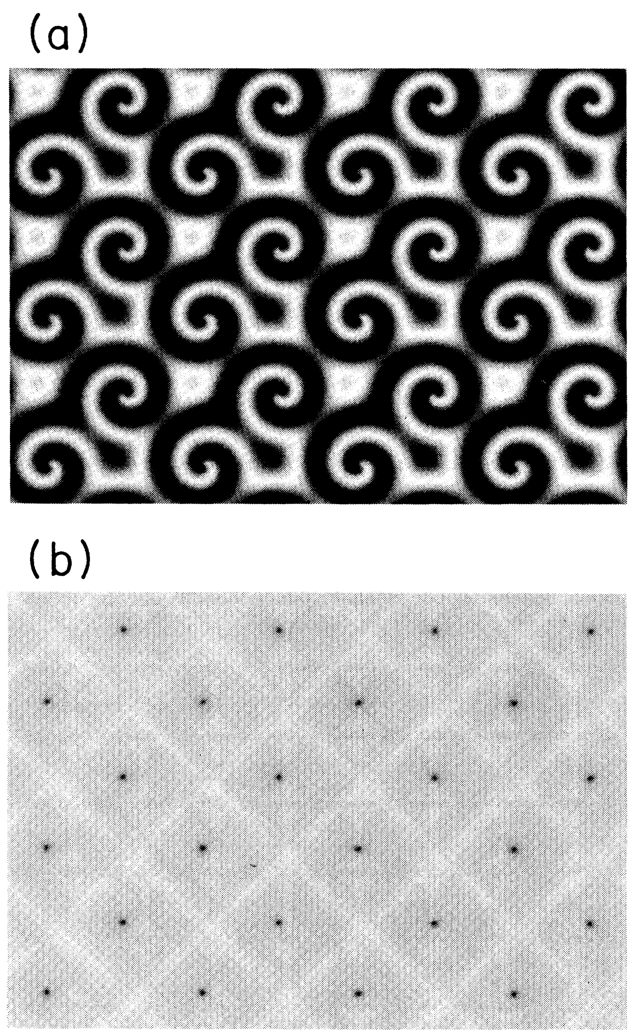

FIG. 3. Stable symmetric lattice of spirals with alternating charge for $b = 0$, $c = 0.7$, $L_x = 400$, $L_y = 300$, periodic boundary conditions. Real part (a) and modulus (b) are shown in gray scale.

where $c' = (c - b)/(1 + bc)$. One sees that the phase difference always tends to zero and the symmetry of the state is restored. Actually, this result can be extracted already from the classical paper of Hagan [10]. From this work one can find the correction to the wave number of a spiral confined to a circular domain. One finds that decreasing the radius increases the wave number, which ultimately means stability of the shocks. The stability of symmetric states, together with the repulsion of the spirals, shows the possibility for the existence of stable symmetric ("antiferromagnetic") lattices of spirals, i.e., lattices made up of free spirals with alternating topological charge. Indeed, such lattices were obtained in numerical simulations in the monotonic range (see Fig. 3). Moreover, we have verified that starting from strongly asymmetric initial conditions leads in the monotonic case eventually to the restoration of the symmetry. We have not observed hysteresis in the symmetry-breaking and restoring transition. However, since the simulations are extremely time consuming, a small hysteresis cannot be excluded.

The results also carry over to the equally charged spiral states. As was shown in [4] for large separation X, the interaction of equally charged spirals is similar to that of oppositely charged ones. The only difference is that for the equally charged case both components of the velocities of the spirals have opposite sign, whereas for the oppositely charged case the tangential components have the same sign. This causes the rotation of equally charged spirals around the common center of symmetry. For $|b - c|$ below the critical value, one has repulsion at large distance (as in the oppositely charged case) and there is repulsion also at a small distance. So it is quite clear that the interaction is repulsive everywhere.

For equally charged spirals one can expect the same mechanism of symmetry breaking as for an oppositely charged pair. Indeed, for $|b - c|$ above the critical value, such a breaking was observed in numerical simulation. Like-charged spirals may also form more complicated bound states or aggregates. In contrast to the two-spiral bound states, which are simply rotating with constant velocity, each spiral in the aggregate performs a more complicated motion on the background of a steady rotation.

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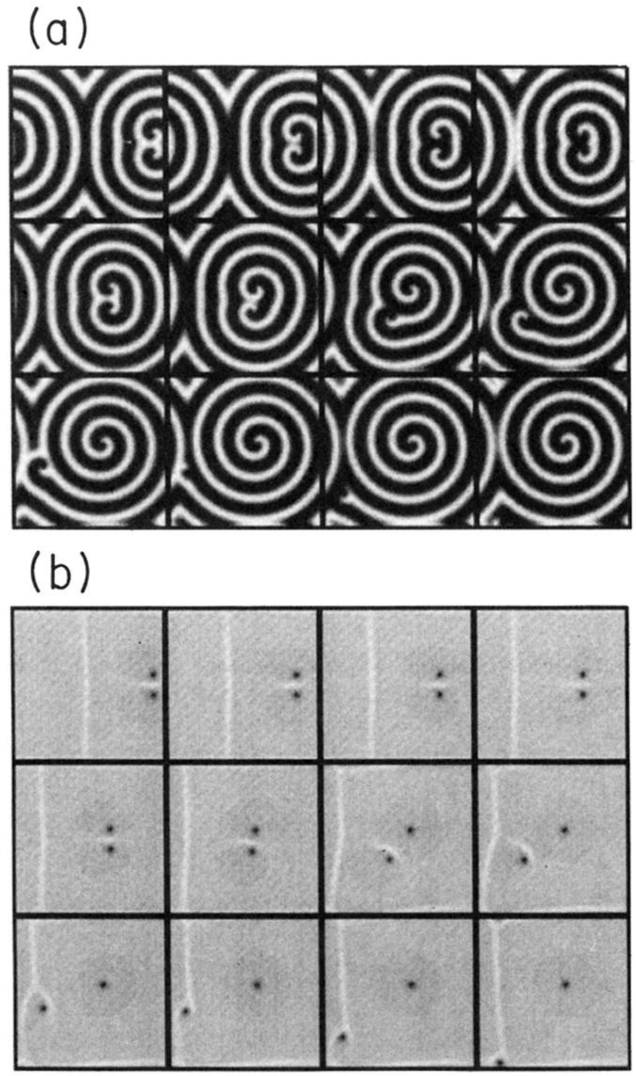

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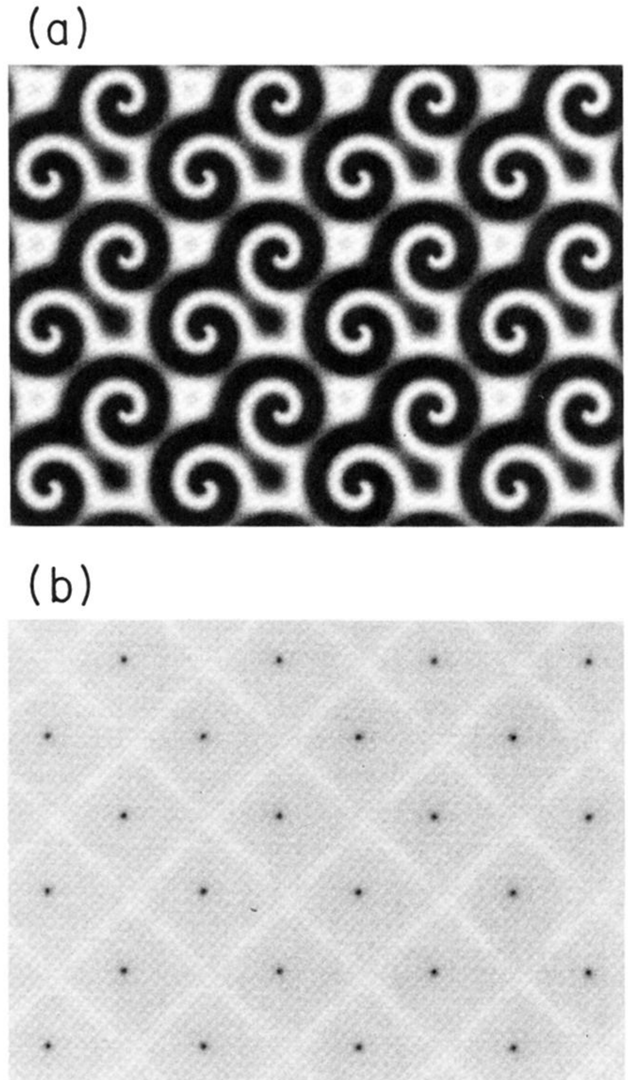

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